Local Stability Analysis of Microwave Oscillators Based on Nyquist's Theorem

Vittorio Rizzoli, *Fellow, IEEE*, Andrea Neri, *Member, IEEE*, and Diego Masotti

***Abstract*—The numerical implementation of Nyquist's theorem for the local stability analysis of forced nonlinear microwave circuits operating in a time-periodic electrical regime is well established. However, the straightforward application of the same algorithm to the stability investigation of autonomous circuits, such as oscillators, may lead to paradoxical results lacking in physical meaning. After demonstrating this by a typical example, we discuss the correct extension of Nyquist's stability analysis to autonomous nonlinear circuits and illustrate the application of the new algorithm to a simple microwave dielectric resonator oscillator (DRO).**

***Index Terms*— Harmonic-balance, nonlinear CAD, stability analysis.**

I. Introduction

The numerical implementation of Nyquist's analysis to investigate the local stability properties of time-periodic regimes in forced nonlinear microwave circuits has been investigated by many authors and is now well established [1]–[4]. On the contrary, the application of the same concept to autonomous circuits such as oscillators is more delicate, and not as well understood. The analysis technique can be summarized as follows. Given a time-periodic regime of a forced nonlinear circuit, a small perturbation of the form $\exp[(\sigma + j\omega)t]$ is superimposed on the steady-state signals, and a first-order perturbative solution of the circuit equations is carried out [2]. The requirement that the perturbation be self-sustained leads to a characteristic equation for the natural frequencies (NF) of the steady state of the form $D(\sigma + j\omega) = 0$, where $D(\cdot)$ is the determinant of a suitable complex matrix [2]. Nyquist's method is then applied in order to establish the number of NF having positive real parts, without explicitly finding the roots of $D(\sigma + j\omega)$. Since $D(\sigma + j\omega)$ is periodic with respect to ω, the Nyquist plot is a bounded closed curve, so that its numerical construction is an easy job [1], [2]. Also, since $D(\sigma - j\omega) = D^*(\sigma + j\omega)$, $D(0)$ is real [2]. This leads us to the crucial point. Whenever a sign reversal of $D(0)$ occurs on the periodic solution path of a parametrized forced circuit, the number of clockwise encirclements of the origin changes by one, which corresponds to the creation or annihilation of a positive real NF, and thus to the loss or recovery of stability with respect to synchronous perturbations [5]. This implies that $D(0) = 0$ is the condition for the occurrence of a direct-type bifurcation (usually a regular turning point) [2], [5]. Such conclusions are no longer valid for an autonomous circuit. In order to show this by a counter-example, let us consider the simple microstrip DRO topology shown in Fig. 1. The field-effect transistor (FET) is a 300-μm device (Alenia S30) described by the nonlinear model reported in [6]. Fig. 2 shows the bifurcation diagram for this circuit parametrized by the distance L between the resonator center and the FET gate. The Hopf bifurcations H_1, H_2 are both subcritical, in the increasing and decreasing parameter sense, respectively, and the only bifurcations encountered on the periodic solution path $H_1T_1T_2H_2$ are the regular turning points T_1, T_2. A dc stability analysis of any of the dc states lying below H_1 shows that all such states are stable. Thus, according to the rules of bifurcation theory, the oscillatory states belonging to the branch T_1T_2 are synchronously stable, while those belonging to the branches H_1T_1, T_2H_2 have one positive real NF, and are thus synchronously unstable [5]. If we now compute the number of unstable NF of the same states by the conventional Nyquist method, we obtain the results shown by dashed lines in Fig. 3. This analysis predicts the occurrence of two unstable NF in some regions of the branch H_1T_1, and a rapid alternation of narrow stable and unstable regions on the branch T_1T_2, in contrast with both bifurcation theory and physical likelihood.

The purpose of this letter is to analyze the reasons for this anomalous behavior and to discuss the correct extension of Nyquist's method to autonomous circuits.

II. Nyquist's Method for Autonomous Steady States

The algorithm for the computation of $D(j\omega)$ is discussed in detail in [2] and only the final result will be reported here. Let n_D be the number of nonlinear device ports, and N be the number of positive harmonics used to describe the steady-state waveforms. Also, let us introduce the complex matrix of dimensions $n_D(2N+1) \times n_D(2N+1)$

$$\mathbf{M}(j\omega) \equiv [\mathbf{Y}(\omega + k\omega_0)\mathbf{P}_{ks} + \mathbf{Q}_{ks}] \qquad (-N \leq k, s \leq N) \tag{1}$$

where $\mathbf{Y}(\omega)$ is the admittance matrix of the linear subnetwork. $\mathbf{P}_{ks}, \mathbf{Q}_{ks}$ are $n_D \times n_D$ submatrices of the nonlinear subnetwork conversion matrices in the neighborhood of a known periodic steady state of fundamental angular frequency ω_0 [2]. On the imaginary axis, $D(j\omega)$ is then given by $\det[\mathbf{M}(j\omega)]$ [2].

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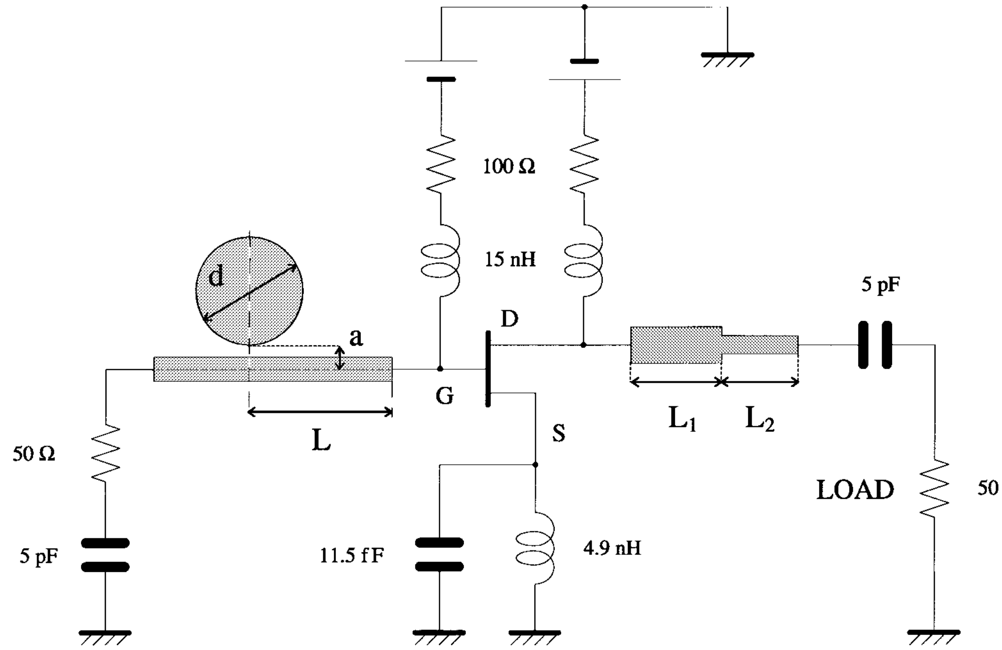

Microstrip lines:	ε_{sub} = 9.7	L_1 = 1.8 mm	L_2 = 1.52 mm	L = free parameter
	h_{sub} = 0.635 mm	W_1 = 1.06 mm	W_2 = 0.81 mm	W = 0.61 mm
Dielectric resonator:	ε_{dr} = 37.28	d = 5.462 mm	tuning plate distance = 2 mm	
	h_{dr} = 2.185 mm	a = 0.92 mm	unloaded Q (10.8 Ghz) = 3600	
MESFET model:	Alenia S30	I_{dss} = 75 mA	V_p = -1.45	

Fig. 1. Schematic topology of a simple microstrip FET DRO.

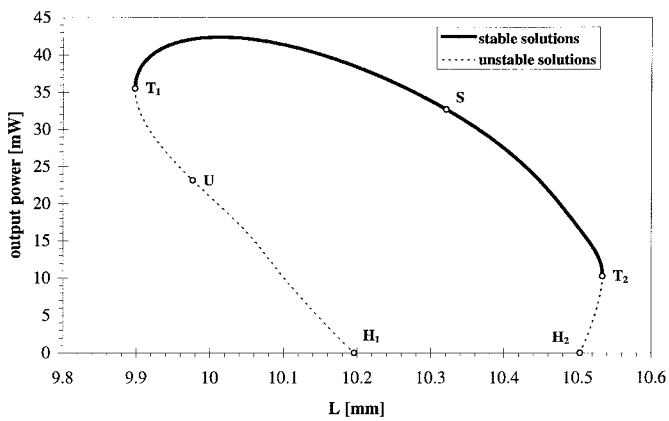

Fig. 2. Bifurcation diagram for the oscillator shown in Fig. 1. The quantity plotted in the ordinate is the output power at ω_0. H_1, H_2 are Hopf bifurcations.

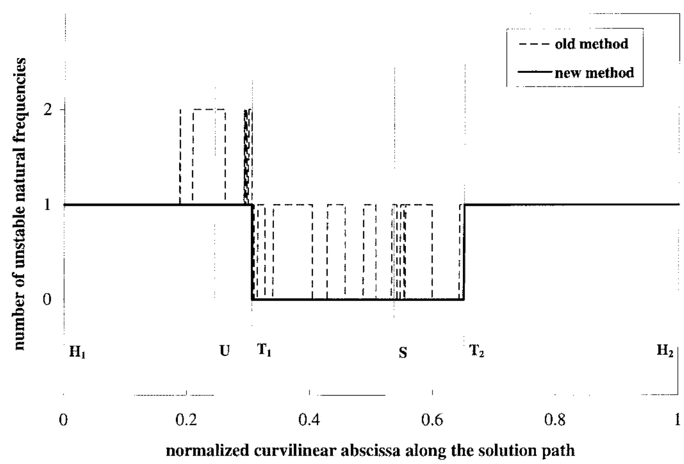

Fig. 3. Number of unstable natural frequencies of the oscillatory regimes belonging to the branches H_1T_1, T_1T_2, T_2H_2. Dashed lines: computed by the Nyquist stability criterion for forced circuits. Solid lines: computed by the modified criterion discussed in this letter.

If we now let $\omega = 0$, the generic submatrix of $\mathbf{M}(j\omega)$ reduces to the corresponding submatrix of the Jacobian matrix of the complex harmonic-balance (HB) errors with respect to the state-variable harmonics [7], so that $\mathbf{M}(0) = \mathbf{J}$. In the case of a forced circuit, this simply confirms the results of the classic analysis, since the Jacobian matrix is known to be singular at a turning point. For an autonomous circuit the phase of the steady state is not determined by the HB equations since there are no sources other than dc. Thus, the HB solving system has ∞^1 time-periodic solutions, and its Jacobian matrix is singular at each autonomous steady state. The conclusion is that for an autonomous regime $D(0) = 0$, i.e., *the origin of the complex plane always belongs to the Nyquist stability plot.* Due to the essential singularity of $D(\sigma + j\omega)$ at infinity, on the complex plane $D(j\omega)$ makes a number of counterclockwise encirclements of the origin, say N_∞, as ω is swept from 0 to ω_0 [2]. N_∞ may be obtained from the Nyquist plot of a stable state, and will thus be regarded as known. Let us now introduce the complex function

$$F(\sigma + j\omega) = \exp\left[-\frac{2\pi N_\infty}{\omega_0}(\sigma + j\omega)\right] D(\sigma + j\omega) \qquad (2)$$

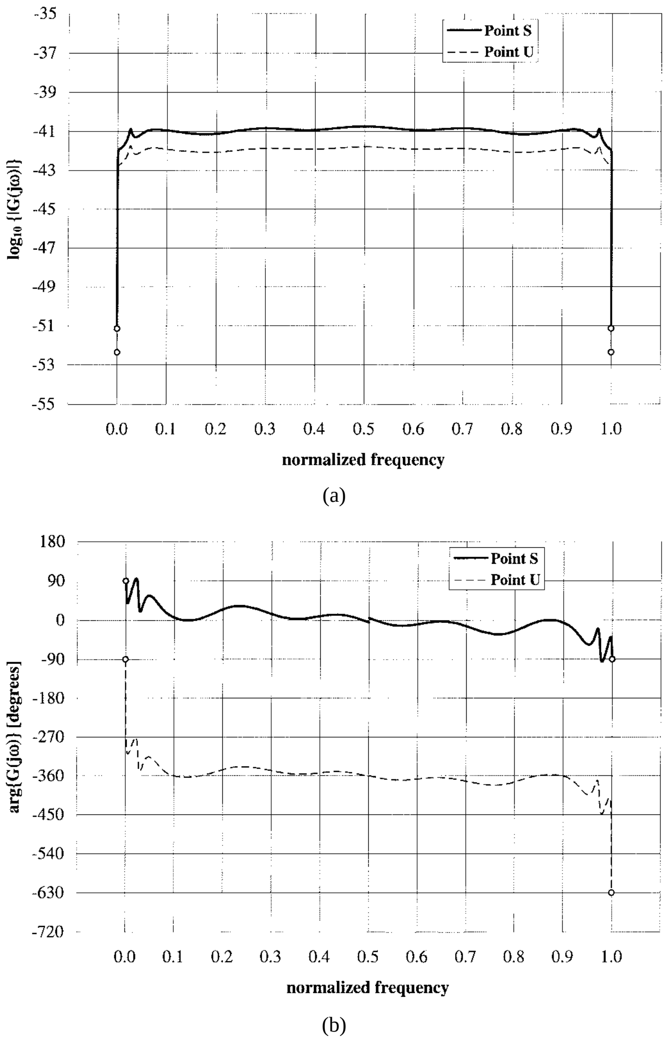

Fig. 4. Nyquist stability plots for the oscillatory states represented by points S and U in Figs. 2 and 3 as a function of normalized frequency ω/ω_0. (a) magnitude; (b) phase in degrees.

where $F(\sigma + j\omega)$ has the same zeros as $D(\sigma + j\omega)$, but at the same time has $N_\infty = 0$. Furthermore, $F(\sigma + j\omega)$ is a periodic function of ω of period ω_0. If we apply Nyquist's theorem to $F(\sigma + j\omega)$ in the range $0 \leq \omega \leq \omega_0$ considering that the origin belongs to the plot, the Nyquist stability equation takes on the form [8]

$$\phi(0) - \phi(\omega_0) \triangleq \Delta\phi = \pi + 2\pi N_Z \tag{3}$$

where $\phi(\omega) = \arg[F(j\omega)]$, $\Delta\phi$ is the total phase change of $F(j\omega)$ around the plot, and N_Z is the number of zeros with positive real parts lying in the frequency range $0 \leq \omega \leq \omega_0$. Since $\Delta\phi$ can be obtained by inspection of the Nyquist plot, N_Z can easily be derived. The autonomous regime is stable if $N_Z = 0$.

The numerical application of the above technique is not as straightforward as it might appear at a first glance. The computation of $D(j\omega)$ is usually carried out by some triangular decomposition method such as Gauss or Crout, which has to be applied to a complex matrix of large order. Due to the unavoidable numerical errors, this procedure will return for $D(0)$ a relatively small but substantially random number instead of zero. Along a solution path such as the one depicted in Fig. 2, $F(0) = D(0)$ will then randomly alternate between positive and negative values, and meaningless stability results such as those plotted by dashed lines in Fig. 3 will be produced. The automatic recognition of this numerical zero is very difficult in practice, because the order of magnitude of $|D(j\omega)|$ varies wildly as a function of ω, of the number of harmonics, and of the free parameter along the solution path. Thus, the safest way of obtaining a stability plot satisfying the theoretical requirements is to apply Nyquist's criterion to the quantity

$$G(\sigma + j\omega) = F(\sigma + j\omega) - D(0)\,. \tag{4}$$

The resulting Nyquist plot exactly contains the origin (to machine accuracy) without being significantly perturbed for $\omega \neq 0$.

As an example of application, Fig. 4 shows the modified Nyquist stability plots (in rectangular coordinates) for the oscillatory states represented by points S and U in Figs. 2 and 3. At point S the plot yields $\Delta\phi = \pi$, so that from (3) $N_Z = 0$. At point U the plot yields $\Delta\phi = 3\pi$, so that from (3) $N_Z = 1$. Thus according to the proposed method, the state S is stable, while U is unstable because of one positive real natural frequency, in agreement with bifurcation theory, but in contrast with the results obtained from the conventional Nyquist analysis (dashed lines in Fig. 3). Finally, the solid lines in Fig. 3 show the number of unstable NF of all the oscillatory states belonging to the periodic solution path of Fig. 2, as computed by the modified Nyquist method. Once again the results exactly agree with those obtained by means of bifurcation theory.

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